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Interface depinning versus absorbing-state phase transitions

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According to recent numerical results from lattice models, the critical exponents of systems with many absorbing states and order parameter coupled to a nondiffusive conserved field coincide with those of the linear interface depinning model within computational accuracy. In this paper the connection between absorbing-state phase transitions and interface pinning in quenched disordered media is investigated. For that, we present an heuristic mapping of the interface dynamics in a disordered medium into a Langevin equation for the active-site density and show that a Reggeon-field-theory-like description, in which the order parameter appears coupled to an additional nondiffusive conserved field, emerges rather naturally. Reciprocally, we construct a mapping from a discrete model belonging in the absorbing state with a conserved-field class to a discrete interface equation, and show how a quenched disorder, typical of the interface representation is originated. We discuss the character of the possible noise terms in both representations, and overview the critical exponent relations. Evidence is provided that, at least for dimensions larger that one, both universality classes are just two different representations of the same underlying physics.

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I. INTRODUCTION

Phase transitions separating a nontrivial from a frozen phase, in which the dynamics is completely arrested, appear in a large variety of situations in physics, as well as in many other disciplines [1–3]. A central problem from a theoretical viewpoint is to understand how the symmetries and conservation laws of the dynamics are reflected in the categorization of models into universality classes. There are two main general contexts in which this type of frozen states appear,

(i) Lattice models with discrete particles; typically particles originate ''activity'' and the frozen state without activity is referred to as ''absorbing state'' [1–3]. This group appears in various disguises as cellular automata [4], reaction-diffusion systems [1,3], directed-percolation-type models [3], or the fixed energy ensemble of sandpile cellular automata [5], among many other examples.

(ii) Elastic interfaces in random environments. In this group, the dynamics is frozen whenever the interface is *pinned* by the quenched disorder, while the nontrivial phase is the moving or depinned one [6,7].

The number of physical realizations of both of these two generic families of phase transitions is huge [1–3,6,7].

The most prototypical universality class in the first group is that embracing, among many other models and systems, directed percolation (DP) [1–4]. At a continuous level the DP class is represented by the Reggeon field theory (RFT) [8], which can be written in terms of the following Langevin equation:

$$\partial_t \rho(\mathbf{x},t) = a\rho - b\rho^2 + \nabla^2\rho + \sigma\sqrt{\rho}\,\eta(\mathbf{x},t), \tag{1}$$

where ρ is an activity field, a, b, and σ are constants and η is a δ-correlated Gaussian white noise. The RFT is the minimal field theory capturing the relevant ingredients of the DP universality class. It can be renormalized using standard field theoretical methods and the associated critical exponents can be computed in ϵ expansion [8]. Other universality classes of absorbing-state phase transitions have been identified; all of them owe their existence to the presence of some additional symmetry or conservation law with respect to the broad DP class. Among them some example are the conserved parity class, in which there are two Z_2-symmetric equivalent absorbing states [9,3], dynamical percolation [10], and the different classes of transitions with extra conservation laws [11–13].

In the group of pinned interfaces, the simplest continuous model for depinning is the quenched Edwards-Wilkinson equation, also called, ''linear interface model'' (LIM) [6,7],

$$\partial_t h(\mathbf{x},t) = \nu\nabla^2 h(\mathbf{x},t) + F + \eta(\mathbf{x},h(\mathbf{x},t)), \tag{2}$$

which describes an elastic interface (the Laplacian) at the reference height $h(\mathbf{x},t)$, with surface tension ν, under the influence of a constant external driving term F, and a *quenched noise* $\eta(\mathbf{x},h(\mathbf{x},t))$. Equation (2) exhibits a *depinning transition* at a critical force F_c; the interface configuration and dynamics develop critical correlations in the vicinity of the critical point. The standard approach for a theoretical analysis of the LIM is the functional renormalization group method. One-loop expressions for the minimal set of exponents have been computed by Nattermann *et al.* [14] on one hand, and by Narayan and Fisher [15] (see also the more recent work by Le Doussal and collaborators [16]). Here one enters technically and conceptually difficult terrain due to the renormalization of the whole disorder correlator. The outcome is that for noise fields η, which do not exhibit extra translational symmetries, the expected depinning behavior follows, very generally, that resulting from a random-field uncorrelated noise term: the LIM universality class [14,15,17,18]. Other universality classes in the interfaces-in-random-media realm are the quenched Kardar-Parisi-Zhang (KPZ) equation [6,7,19] and the Edwards-Wilkinson equation with columnar noise [7,20].

TABLE I. Critical exponents for steady-state experiments in $d=2$ and $d=3$. Models: CTTP, conserved threshold transfer process; CRD, conserved reaction-diffusion model; Manna, Abelian Manna sandpile; LIM, linear interface model. See the Appendix for exponent definitions.

	β	$\nu_\perp$	$\beta/\nu_\perp$	z
		Steady-state exponents $d=2$		
CTTP	0.64(1)	0.82(3)	0.78(3)	1.55(5)
CRD	0.65(1)	0.83(3)	0.78(2)	1.55(5)
Manna	0.64(1)	0.82(3)	0.78(2)	1.57(4)
LIM	0.64(2)	0.80(1)	0.80(3)	1.56(6)
		Steady-state exponents $d=3$		
CRD	0.86(2)	0.63(5)	1.39(4)	1.80(5)
Manna	0.84(2)	0.60(3)	1.40(2)	1.80(5)
LIM	0.84(2)	0.606(4)	1.38(2)	1.75(15)

Recent investigations (motivated by the analysis of sandpile models [21,22], the archetype of systems exhibiting self-organized criticality (SOC) [23]) have demonstrated that different models showing a continuous transition into an absorbing phase and with an order parameter coupled linearly to an extra, nondiffusive conserved field (NDCF) belong to a unique universality class [13,24,25] that we will refer to as NDCF class. This class differs from the extremely robust DP class owing to the presence of an additional conservation law [12]. Moreover, the critical exponents of this class seem, within numerical accuracy, equal to those of the LIM class [5,13,24,25]. In Table I we present a comparison of numerical results obtained for: (1) two different models with a nondiffusive conserved field and many absorbing states, namely the conserved threshold transfer process (CTTR) [13,24], and a conserved reaction-diffusion model (CRD) [13,24], (2) the fixed energy version of the Manna sandpile model and, (3) the LIM model. Observe that all the reported exponents coincide within numerical accuracy in both $d=2$ and $d=3$. This might be surprising at first sight, as in CTTP and CRD models there is no quenched disorder, as there is in LIM, and quenched disorder is usually a relevant perturbation when it comes to universality issues.

From a different perspective this observation is not so surprising, as different tentatives have been reported in the literature in order to relate the dynamics of sandpiles to that of elastic manifolds in random media, i.e., to the LIM model [26,27], and also the same sandpiles have been argued to belong to the NDCF class [5,25]. Furthermore, there is another viewpoint from which the coincidence between both types of models is not so striking, namely, that provided by the ''run-time statistics'' theory [28]. This theory establishes that quenched disorder can be mapped rather generically into long-range temporal correlations (i.e., a long-term memory) in the activity field, (note this relation works also the other way around) [30], and has been recently applied with success to the Bak-Sneppen model among others [29]. In the NDCF class the presence of a conserved field plays the role of a long-term memory [5,25] and, therefore, it is not a very big surprise that it is equivalent to some sort of quenched disorder.

In this article we discuss in detail the relation between the two presented groups of transitions, i.e., absorbing states with a conserved field and pinned interfaces in random media, including annealed (or thermal) noise and quenched disorder, respectively. We will present heuristic arguments providing a theoretical explanation for so different systems sharing the same universal critical behavior. The connection between absorbing-state models in the DP class (without a conserved field) and their interface representation has also been recently considered in the literature [31]. In particular the RFT was mapped into rather unusual interface equation, not resembling any known interfacial problem.

The paper is structured as follows: We start in Sec. II by presenting the RFT-like Langevin equation for the recently introduced NDCF class. In Sec. III we present a prototypical interface model in the LIM class, in particular, the cellular automaton by Leschhorn [17] (see also [18]) and work out a derivation of a Langevin equation for the activity density ρ, paying particular attention to the way by which the noise can be found. In Sec. IV we proceed conversely: we employ a discrete mapping of a model with absorbing states in the NDCF class into a continuous interface representation. We end up with an interface equation, with several quenched noise terms that reflect the microscopic rules and the thermal noise applied in them. We discuss at this point the noise correlations that arise and their relevance, with the aid of the renormalization group (RG) literature. Finally, we present a discussion and an Appendix in which we outline the relations between the exponents in the two different pictures.

II. THE NDCF FIELD THEORY

One particular system in the NDCF class (out of the many studied [13,24]) is a two-species reaction-diffusion model, in which one of the species is immobile [11] (see Sec. IV for a detailed definition). It has the great advantage of allowing for a rigorous derivation of a coarse-grained field theory (or, equivalently, a Langevin equation) via a Fock space representation of the dynamics [11,24,32]. The result is in the form of a Reggeon field theory coupled to an extra conserved nondiffusive field, or what is equivalent, a RFT equation with an extra non-Markovian term [5,24,25]. Quite remarkably this Langevin equation coincides (up to irrelevant terms) with the one proposed previously, based only on symmetry and relevancy arguments, as the minimal Langevin

equation capturing the physics of NDCF, namely [5,24],

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} = a\rho(\mathbf{x}) - b\rho(\mathbf{x})^2 + \nabla^2 \rho(\mathbf{x},t) - \mu \psi(\mathbf{x},t)\rho(\mathbf{x},t) + \sigma\sqrt{\rho(\mathbf{x},t)}\,\eta(\mathbf{x},t),$$

$$\frac{\partial \psi(\mathbf{x},t)}{\partial t} = D\nabla^2 \rho(\mathbf{x},t), \tag{3}$$

plus higher-order terms, irrelevant from naive power counting analysis [33]. Note that the second equation, describing the evolution of the background conserved field (coarse-grained representation of the total number of particles, which is conserved in the microscopic model), represents an static nondiffusive field: in the absence of activity its dynamics is frozen. Observe also that the second equation, being linear, can be integrated out, and a closed equation for the activity written down. More concretely,

$$\psi(\mathbf{x},t) = \psi(\mathbf{x},0) + D\int_0^t dt' \nabla^2 \rho(\mathbf{x},t'). \tag{4}$$

The first contribution in Eq. (4), a quenched (columnar) disorder, represents the initial condition, while the second is a non-Markovian term. The Langevin equation (3), even though it looks rather similar to the RFT, has resisted all renormalization attempts; therefore, predictions about critical exponents coming from an epsilon expansion calculation are not available so far. This might be an indication that some type of *functional* renormalization group calculation is required, as is the case for the LIM equation, but this issue needs certainly further insights to be clarified.

III. PHENOMENOLOGICAL ACTIVITY DESCRIPTION OF LIM MODELS

We consider a representative of the LIM class, namely, the Leschhorn-Tang (LT) cellular automation [17]. In order to study its relation with standard systems with absorbing states, we intend to cast it into a Langevin equation describing the evolution of a coarse-grained activity field [2].

The LT automation is defined as follows. The interface field $h(\mathbf{x})$ satisfies at each discrete time step t_i the following equation:

$$h(\mathbf{x},t_{i+1}) = \begin{cases} h(\mathbf{x},t_i)+1, & f(\mathbf{x},t_i) > 0 \\ h(\mathbf{x},t_i), & f(\mathbf{x},t_i) \leqslant 0, \end{cases} \tag{5}$$

where the force f is given by the combination of elasticity and a random quenched pinning force as

$$f(\mathbf{x},t_i) = \nabla^2 h(\mathbf{x},t_i) + \eta(\mathbf{x},h). \tag{6}$$

$\nabla^2 h(\mathbf{x})$ is the discrete Laplacian, i.e., $\Sigma_{\mathrm{NN}} h(\mathrm{NN}) - 2Dh(\mathbf{x})$ and NN denotes the nearest neighbors on a hypercubic lattice. A reasonable choice for the noise is

$$\eta(\mathbf{x},h) = \begin{cases} +1, & p \\ -1, & 1-p \end{cases} \tag{7}$$

when p is a random number uniformly distributed between zero and unity. This choice implies that the average driving force is $F = \langle f \rangle = 2p - 1$. F plays the role of a control parameter. The critical point is estimated to be at $p_c \sim 0.800$ [17].

Now, at every time step, and at each site where the total driving force exceeds its threshold value, i.e., at each interface-site advance, we define an activity variable and set it equal to one. On the other hand, in the remaining lattice sites the corresponding activity takes a zero value. Additionally, we also define at each site and time, a continuous ''background'' variable, equal to $\nabla^2 h(\mathbf{x},t) + F$. This controls the probability of each interface site to advance at each time, regardless of whether it actually slips or not. Let us emphasize that this background variable is a conserved magnitude, i.e., it takes a constant value, equal to F, when integrated (summed) over the whole lattice. However, locally, it favors or inhibits the generation of new activity. We now build up a couple of equations for the evolution of the two fields: the activity, $\rho(\mathbf{x},t)$, and the background field, $\psi(\mathbf{x},t)$, which are the coarse-grained field analogous of the previously defined site variables. Using the identification between activity and ready-to-advance sites: $h(\mathbf{x},t) = \int_0^t dt' \rho(\mathbf{x},t') + h(\mathbf{x},0)$. Let us write down a couple of *mean-field* equations for the two defined fields:

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} = -\rho(\mathbf{x},t) + \rho(\mathbf{x},t)\mathcal{G}[\psi(\mathbf{x},t)]\nabla^2 \rho(\mathbf{x},t), \tag{8}$$

$$\psi(\mathbf{x},t) \equiv \nabla^2 h(\mathbf{x},t) + F = \int_0^t dt \nabla^2 \rho(\mathbf{x},t) + \nabla^2 h(\mathbf{x},0) + F, \tag{9}$$

where $\mathcal{G}$ is an unknown functional of the background field. The justification of the different terms is as follows:

(i) The term—$\rho(\mathbf{x},t)$ describes the decay of active sites, that after the corresponding interface advance become, in general, nonactive. At a coarse-grained level higher-order corrections, as $-b\rho^2(\mathbf{x},t)$ may also appear. In particular, they might play an important role in order to prevent the activity from growing unboundedly, i.e., in stabilizing the theory.

(ii) $+\rho(\mathbf{x},t)\mathcal{G}[\psi(\mathbf{x},t)]$ represents the fact that activity is created in regions where some activity is already present, and the rate of creation at each point is a function of the local background field, $\psi(\mathbf{x},t)$. Observe that the total contribution of this term when integrated over the whole space has to be zero, but locally it fosters or inhibits the creation of further activity. Again, higher-order powers of $\rho(\mathbf{x},t)$ might also be included.

(iii) $\nabla^2 \rho(\mathbf{x},t)$ describes the diffusion of activity. This terms appears generically for diffusive systems at a coarse-grained scale.

(iv) In what respects the $\psi(\mathbf{x},t)$ field, Eq. (9), we have just written its definition by equating $h(\mathbf{x},t)$ to the number of ''topplings'' (or activity events) at that point in all the preceding history, plus its initial value.

Expanding $\mathcal{G}[\psi(\mathbf{x},t)]$ in power series, and keeping only the leading contribution, we are left with a term $+\lambda\rho(\mathbf{x},t)\psi(\mathbf{x},t)$ (where λ is a constant) on the right-hand side of Eq. (8) (observe that the constant term in the Taylor

expansion has to be zero as its integral has to be conserved, as argued before). *A posteriori*, we shall show that the omitted terms, as well as higher-order corrections to the Laplacian term, are irrelevant in what respects large scale, asymptotic, properties.

In order to account for the system fluctuations (completely ignored so far) we now introduce a noise field contribution to Eq. (8). For that, as it is well known in theoretical field descriptions of systems with absorbing states [2,3], a RFT noise term: $\sigma\sqrt{\rho(\mathbf{x},t)}\,\eta(\mathbf{x},t)$ is needed, where σ is a constant and η a Gaussian white noise. This just reflects the fact that, as ρ is a local coarse-grained variable its local fluctuations are proportional to its square root (see [1–3] and references therein). It also captures the physical key ingredient: wherever activity vanishes locally, fluctuations are canceled [2].

Before proceeding further, let us now discuss why the quenched disorder of the microscopic model can be represented by an annealed noise in the derivation shown before. The key point is the observation that in active regions, i.e., where the interface advances, a new noise variable is selected at every time step and, as the interface does not return to already passed regions, there is no need to store the microscopic noise history, and the noise can be freshly extracted from its probability distribution after every interface advance. In this way, it becomes rather obvious that in depinned (active) regions, quenched and annealed noises are fully equivalent at the microscopic level. Upon coarse graining, the noise can be expected to acquire a RFT-like character as the correlations in the integrated activity, or number of simultaneously active sites vanish on large enough scales. More subtle is the connection between the two types of noises with respect to pinned (absorbing) regions. While the annealed noise, η changes in time even if there is no activity in a given region, its variations are completely irrelevant as the noise amplitude appears multiplied by $\sqrt{\rho}=0$. Noise (including its activity dependent amplitude) at a given spatial point changes only whenever activity arrives to it, mimicking perfectly what happens in the microscopic interface model, where regions pinned under the influence of an unfavorable quenched noise can be depinned only under the presence of neighboring moving regions. Therefore, the considered time-dependent noise, reproduces properly (at least qualitatively) all the properties of the original quenched disorder.

The previous considerations lead finally to the following Langevin equation for the activity field:

$$\frac{\partial\rho(\mathbf{x},t)}{\partial t}=[-1+\lambda F+\lambda\nabla^2 h(\mathbf{x},0)]\rho(\mathbf{x},t)+\nabla^2\rho(\mathbf{x},t)+\lambda\rho(\mathbf{x},t)\int_0^t dt'\nabla^2\rho(\mathbf{x},t')+\sigma\sqrt{\rho(\mathbf{x},t)}\,\eta(\mathbf{x},t), \tag{10}$$

where we have substituted ψ by its expression coming from Eq. (9). In general, the system is expected to lose memory of the initial state for long enough times, therefore, the dependence on $\nabla^2 h(\mathbf{x},0)$ is expected to be washed out. However, in some cases, as for instance one-dimensional (1D) systems, due to the meager phase space, and the slow relaxation of the initial condition, this might not be the case [34].

Performing a perturbative, diagrammatic study of the previous Langevin equation it is easy to see (already at one-loop level) that a new nonlinearity (vertex), with the same degree of relevancy as the nonlinear terms already present in the theory (i.e., the nonlocal-in-time vertex and the noise one) is perturbatively generated: $\rho^2(\mathbf{x},t)$. In fact, this term could have been introduced also at a mean-field level, as pointed out before, as a stabilizing term for the activity equation.

Including all the discussed terms into the equation for ρ, and integrating the equation for ψ, we finally obtain

$$\frac{\partial\rho(\mathbf{x},t)}{\partial t}=-a\rho(\mathbf{x},t)-b\rho(\mathbf{x},t)^2+\lambda\rho(\mathbf{x},t)\int_0^t dt'\rho(\mathbf{x},t')+\lambda\nabla^2 h(\mathbf{x},0)+\nabla^2\rho(\mathbf{x},t)+\sigma\sqrt{\rho(\mathbf{x},t)}\,\eta(\mathbf{x},t), \tag{11}$$

where $a=-1+F\lambda$ and $b>0$ are constants. At this point, it is a rather straightforward exercise to verify that no further relevant terms are generated when including perturbative (diagrammatic) corrections to the bare theory. Therefore, the resulting Langevin equation is identical to the one proposed for systems with an infinite number of absorbing states and an activity field coupled to a static conserved field Eq. (3) [5,13,24].

Summing up, we have mapped a microscopic model belonging in the LIM class to the Langevin equation characterizing the NDCF class. Though our derivation is not rigorous, we believe it provides a strong evidence that in fact LIM and NDCF define the same universality class.

IV. MAPPING A REACTION-DIFFUSION MODEL TO DEPINNING

In this section we proceed conversely to the previous one: starting from a microscopic model in the NDCF class we map it onto the LIM continuous equation, Eq. (2). To that end we follow a recipe already applied to many sandpile models exhibiting SOC [27]. Following [24] we consider a two-species reaction-diffusion process on a L^d lattice, with particles of types A and B involved. At each site i, and at each (discrete) time step the following reactions take place:

$$B_i\rightarrow B_{\mathrm{NN}},\quad r_d\equiv 1, \tag{12}$$

$$A_i+B_i\rightarrow 2B_i,\quad r_1, \tag{13}$$

$$B_i\rightarrow A_i,\quad r_2. \tag{14}$$

The A_i, B_i denote particles of each kind at site i. The r's are the probabilities for the microscopic processes to occur: diffusion r_d, activation r_1, and passivation r_2. Without loss of generality we will fix $r_d=1$, implying that, after having the chance to react, B particles diffuse with probability one. Thus one can define a phase boundary between the active and absorbing phases in terms of the r_1, r_2 probabilities, with a phase transition in between. We assign occupation numbers $n_{A,i}$, $n_{B,i}$ to each site. As the A particles are nondiffusive,

this system has an infinite amount of absorbing states defined by $n_{B,i}=0$ for all i, with $n_{A,i}$ arbitrary.

Now we define (analogously to what is done for sandpiles [5,27]) a height field $H(\mathbf{x},t)$ which increases by one unit every time a site gives one (or more than one) active, diffusing B particle to one (or more than one) of its neighbors. When this happens, we say, using the sandpile terminology, that the site ''topples.'' In this way, the H field measures the integrated activity at $\mathbf{x}$ up to time t.

The mapping to an interface automaton with quenched noise is based on the fact that both, the reactions between the A and B species, and the diffusion of particles can be accounted for by looking at their net effects at every time B particles leave the site $\mathbf{x}$. One just has to look at n_A and n_B when the site becomes active and a particle diffuses out. The dynamics of H can be written as

$$H(\mathbf{x},t+1)=\begin{cases} H(\mathbf{x},t)+1, & f(\mathbf{x},H)>0 \\ H(\mathbf{x},t), & f(\mathbf{x},H)\leqslant 0, \end{cases} \tag{15}$$

which is formally identical to the Leschhorn automaton in the LIM class, with a local ''force'' defined as

$$f(\mathbf{x},H)=n_{\mathrm{tot}}(\mathbf{x},H)-\xi(\mathbf{x},H), \tag{16}$$

where $n_{\mathrm{tot}}(\mathbf{x},H)=n_A(\mathbf{x},H)+n_B(\mathbf{x},H)$ is the total number of particles at $\mathbf{x}$, and ξ is a local random threshold that results from the microscopic processes. More concretely, the noise ξ is defined as follows: Consider the site $\mathbf{x}$ after the Hth toppling, either $n_B(\mathbf{x},H)=0$ or $n_B(\mathbf{x},H)>0$ (this last can be the case if and only if particles have arrived from the nearest neighbors at the same time step). In the first case, it will remain zero until a particle arrives from a nearest neighbor site; then one is free to choose a value for $\xi(\mathbf{x},H)$ such that it makes the force f negative in the time interval between topplings H and $H+1$. In the second case, $n_B(\mathbf{x})$ will fluctuate owing to the microscopic passivation and activation processes, either going to $n_B(\mathbf{x})=0$ or inducing a toppling at the next time step. The relative probabilities of these two alternatives, as derived from the microscopic dynamics, are captured in the $\xi(\mathbf{x},H)$ probability distribution.

Observe that ξ depends solely upon the total number of particles after the preceding toppling and the microscopic dynamical rules. In particular, the larger n_{tot} the larger the probability to have many B particles and the larger the probability to topple. Let us also remark that the immobile grains n_A constitute a ''pinning force'' (the larger their relative number, the lesser the probability to topple). The point-wise noise field $\xi(\mathbf{x},H)$ should have two-point weak correlations in $\mathbf{x}$ since, in particular, it depends on the number of grains received from the NN's at the interface location $H(\mathbf{x})$ which induces weak site-site correlations. The fact that n_A changes slowly will make the H part of the noise correlator $\langle\xi(\mathbf{x},H)\xi(\mathbf{x}',H')\rangle$ less trivial than a simple delta function $\delta(H-H')$.

Equation (15) can be considered as a discrete interface equation

$$\frac{\Delta H}{\Delta t}=\theta(f(x,t)). \tag{17}$$

It can be rewritten with the help of two particle fluxes: $n_{\mathbf{x}}^{\mathrm{in}}$ and $n_{\mathbf{x}}^{\mathrm{out}}$, are the number of grains added to or removed from a given site $\mathbf{x}$ up to time t, respectively. Let us also define g as the average number of particles given to the nearest neighbors at each toppling event. It is clear that for long enough times $n_{\mathbf{x}}^{\mathrm{out}}\approx gH(\mathbf{x})$; relative deviations from this equality being negligible asymptotically. Defining also the average value of $n_{\mathbf{x}}^{\mathrm{in}}$, $\bar{n}_{\mathbf{x}}^{\mathrm{in}}$, as $\bar{n}_{\mathbf{x}}^{\mathrm{in}}=g/2d\Sigma_{\mathbf{x}_{\mathrm{NN}}}H(\mathbf{x}_{\mathrm{NN}},t)$, we can compute a noise $\tau(\mathbf{x},H)$ as the deviation of $n_{\mathbf{x}}^{\mathrm{in}}$ with respect to its average value,

$$\tau(\mathbf{x},t)=n_{\mathbf{x}}^{\mathrm{in}}-\frac{g}{2d}\sum_{x_{\mathrm{NN}}}H(\mathbf{x}_{\mathrm{NN}},t). \tag{18}$$

In other words, $\tau(\mathbf{x},t)$ counts the relative proportion of particles diffused out from the neighbors that actually arrive to the site under consideration, compared with its average value. A site to which particles have toppled in excess will take a positive value of τ, and, therefore, will be more likely to topple in the following time steps. Notice that the construction that yields τ is exact.

Plugging this into Eq. (16), and using that $n_{\mathrm{tot}}(\mathbf{x},t)=n_{\mathrm{tot}}(\mathbf{x},0)+n_{\mathbf{x}}^{\mathrm{in}}-n_{\mathbf{x}}^{\mathrm{out}}$, we can write [27]

$$f=\frac{g}{2d}\nabla^2H+F(\mathbf{x},0)-\xi(\mathbf{x},H)+\tau(\mathbf{x},H), \tag{19}$$

where $F(\mathbf{x},0)\equiv n_{\mathrm{tot}}(\mathbf{x},t=0)$.

The discretization in Eq. (17) can be understood so that the rules result in an *effective* force f' that is exactly unity when the interface field H advances. Thus $\Delta H/\Delta t\equiv f'\theta(f)=f'\theta(f')$ [27]. This construction can be achieved by picking ξ to have exactly the right value in order to make the force driving the interface equal to unity, if it is larger than zero. One arrives finally at the discretized interface equation,

$$\frac{\Delta H}{\Delta t}=\frac{g}{2d}\nabla^2H+F(\mathbf{x},0)-\xi(\mathbf{x},H)+\tau(\mathbf{x},H). \tag{20}$$

Let us stress the presence of three different noise terms

(1) $F(\mathbf{x},0)$ represents the original configuration of total number of particle at $t=0$, and is, therefore, a *columnar noise* term [20]. It induces an initial transient regime until eventually, the dynamics washes out the dependence of the original configuration. In general, columnar disorder is irrelevant in the renormalization group sense as compared to quenched noise; therefore, using relevancy arguments, it could be eliminated, at least in high enough dimensions, close or above the critical one $d_c=4$. Notice that this statement is equivalent to the LIM symmetry, by which static force fields $F(\mathbf{x},0)$ (independent of H) is completely equivalent to the existence of a nontrivial initial interface profile $H(\mathbf{x},t=0)$. However, in low-dimensional systems, and in particular in $d=1$, due to the meager phase space, relaxation

times might be huge, and the time needed to eliminate the dependence on the initial particle distribution divergently large [34].

(2) The noise term $\xi(\mathbf{x},H)$ represents the local threshold, determining whether a site with some B particles topples at a given time or, alternatively, they are transformed into A particles by microscopic processes. It captures the in-site microscopic dynamics, and depends essentially on $n_{\rm tot}$, and on the microscopic probabilities. In a nutshell, it says how many of the $n_{\rm tot}$ particles are of type A after the microscopic dynamics has operated in the corresponding time step: if all $n_{\rm tot}$ are of type A then $\xi > n_{\rm tot}$, and $f<0$; conversely, if any of the particles is of type B then $\xi < n_{\rm tot}$ and $f>0$. Observe that if the diffusion probability was smaller than unity, then we should substitute $\xi(\mathbf{x},H)$ by a ''thermal noise'' $\xi(\mathbf{x},t)$, i.e., ξ would change its value after every time step instead of changing only after each toppling: this is due to the fact that if $r_d<1$ then a site $\mathbf{x}$ including B-type particles could not topple at time t [$\xi(\mathbf{x},t)$ below threshold], and do so at a future time t' [$\xi(\mathbf{x},t')$ above threshold]. This ''thermal noise'' would generate a transition rounding off, but the critical exponents should not be affected by this irrelevant perturbation [14]. Therefore, we stick to the simplest case $r_d=1$.

(3) The noise term τ keeps track of the Brownian motion of particles; i.e., it takes into account the fact that particles are not homogeneously distributed among the NN, but one of them is picked up randomly for each toppling event. It changes slowly since the effect of the random choices (directions) on the configuration is slow. This is in particular true since the noise τ is conserving, as the number of particles is conserved [and as can be seen by integrating Eq. (18)]. A key point is that, analogously to what discussed in the preceding section, the choice to give a particle to a certain neighbor can be taken to be ''quenched,'' i.e., chosen in advance at $t=0$, or ''annealed,'' i.e., decided on the spot. The correlator of τ can be generically written as

$$\langle \tau(\mathbf{x}',H')\,\tau(\mathbf{x},H)\rangle \sim f_{\parallel}(\mathbf{x}'-\mathbf{x})\,f_{\perp}(H'-H). \qquad (21)$$

The (so far unknown) correlators $f_{\parallel}$ and $f_{\perp}$ reflect the discrete nature of the choices in the dynamics. Two microscopic reasons lead immediately to nontrivial correlations in τ:

(a) The noises τ at the NN's of site $\mathbf{x}$ are correlated due to an exclusion effect: If a site gives out a diffusing B particle to a neighbor, then all the other neighbors are excluded. The actual coarse-grained noise correlations are harder to assess, since the fluctuations in the particle flux that τ measures make the interface to fluctuate, and thus a separable noise correlator as Eq. (21) is hard to compute. The easiest way to analyze the correlations among the different sites is, therefore, to compute the noise correlator from numerics of the microscopic model, using the noise definition Eq. (18). This program has been pursued for sandpiles [27].

(b) At each site the noise follows the dynamics of a random walker. In fact, every time a nearest neighbor topples, the choice (give the particle to $\mathbf{x}$ or to a different site) makes it so that $f_{\perp}\sim(H'-H)^{1/2}$ since at every step τ can go ''up'' or ''down'' with respect to the average.

Therefore, reciprocally to what was done in the preceding section, we have mapped the reaction-diffusion process into an interface equation. The dynamics of this interface equation follows exactly the history of a reaction diffusion process, the details of which are mapped into the quenched noises ξ and τ, and a columnar noise $F(\mathbf{x},0)$. Let us remark that the existence of a conservation law has played a key role in order to obtain a Laplacian in Eq. (20).

Finally, using standard renormalization group arguments about the relevancy of different operators, we can eliminate higher-order irrelevant terms and noise correlations, and then we are left with the LIM equation for point disorder [14,16] (see also the Appendix).

It must be emphasized, that the mapping works in both ways, it is evident that the noise construction can be inverted to yield a reaction diffusion process, that corresponds to an interface model, assuming that the original noise terms have the right correlation and conservation properties. The interface model Eq. (20) resembles very much the one that corresponds to the Manna sandpile automaton, with the addition of the ξ-noise term that is more point disorder like than the τ term.

Summing up, reciprocally to what was done in the preceding section, in this one, we have constructed a mapping between a microscopic model in th NDCF class into the Langevin equation for the LIM class.

V. DISCUSSION

We have presented strong heuristic evidence that, rather generically, the universality class of systems with many absorbing states and order parameter coupled to a nondiffusive conserved field, the NDCF class, and that of the linear interface model with point-disorder coincide. This fact, already pointed out from numerical simulations [5,13,24] is true at least nearby the critical dimension $d_c=4$, where relevancy arguments are reliable. In low-dimensional systems ($d=1$) this equivalence could break down owing to the existence, for example, of slow decaying initial conditions [34]. For the frozen configurations in the point-disorder LIM it is known that the correlations of the forces $\eta(\mathbf{x},H)$ acting on the interface vanish. In the case of NDCF models, like the Manna sandpile, such correlations (now computed from the particle configuration in frozen configurations) may become nonzero: this is a future avenue for numerical studies, but hopefully this would be a irrelevant feature.

Likewise, if one considers a noise field for the LIM [Eq. (7)] with nontrivial (power law) bare correlations in x or H, it is unclear at this point how these should be reflected in the construction of a Langevin equation for the corresponding activity field, like Eq. (10). Correlations in the local forces (or ''activity thresholds'') will affect the way the coarse graining works. For instance, due to the noise structure the pinned and still-active regions will be correlated.

In order to establish the connection between the two classes we have mapped a discrete interface model into the Langevin equation characteristic of the NDCF class, and conversely mapped a discrete model in the NDCF into the well-known Langevin equation describing the LIM class. In

order to have a more rigorous proof, one should be able to map one Langevin equation into the other, but this, being the Langevin equations coarse-grained representations of the microscopic models, is not an easy task to fulfill, and remains an open challenge.

Let us remark that a similar problem remains also open; namely, the rigorous connection between the quenched KPZ [6,19] depinning transition and directed percolation depinning [7,36] in two-dimensional systems, (and to directed surfaces in higher dimensions [37]). It is clear from numerics, that indeed these two universality classes coincide, but a satisfactory proof of this fact is, to the best of our knowledge, still lacking.

It was the hope, that the possibility of renormalizing the NDCF Langevin equation using standard RG techniques, of problem from the RFT-like equation approach, could shed some light on the (in principle, technically more difficult and obscure) functional renormalization group analysis required for the interface equation with quenched noise. However, the difficulties encountered in renormalizing, using standard perturbative schemes, the Langevin equation for NDCF [24,25] are considerable; and have made all the attempts to renormalize the theory to fall through. It is rather likely that the failure of standard RG attempts implies that a functional RG scheme is needed in order to properly renormalize the theory, analogously to what happens for the LIM equation. Renormalizing the NDCF Langevin equation and relating the derived critical exponents to those obtained using functional RG for LIM remains an open and very challenging problem.

Finally let us also point out that all the discussions presented in this work deal with the ''constant force'' (in the interface language) or ''fixed energy'' (in the absorbing-state terminology) ensemble. They can be easily extended to the ''constant force'' or ''slow driving'' ensemble [5,27], in which the system self-organizes into its critical state. This point is, however, not essential since all evidence points to the fact that if two models belong to the the universality class, they continue to share the same set of critical exponents upon changing ensemble.

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APPENDIX

The scaling of the phase transition in the absorbing-state representation is characterized by the exponents $\nu_\perp$, $\nu_\parallel$, z, and β. These describe the correlations in the activity ρ in the spatial and time directions, the development of the correlations in time, and the behavior of ρ above the critical point, respectively. One has the scaling relation

$$\overline{\rho}(\Delta,L)=L^{-\beta/\nu_\perp}\mathcal{R}(L^{1/\nu_\perp}\Delta), \tag{A1}$$

where Δ is the distance to the critical point, and $\mathcal{R}$ is a scaling function with $\mathcal{R}(x)\sim x^\beta$ for large **x**. For $L\gg\xi\sim\Delta^{\nu_\perp}$ we expect $\overline{\rho_a}\sim\Delta^\beta$ (here ξ is the correlation length). When $\Delta=0$ we have that $\overline{\rho_a}(0,L)\sim L^{-\beta/\nu_\perp}$. For $\Delta>0$, by contrast, $\overline{\rho_a}$ approaches a stationary value, while for $\Delta<0$ it falls off as L^{-d}. These can be used to establish the numerical values of the exponents.

In the interface representation the relevant exponents are ν, z as above, with the convention that $\nu\equiv\nu_\parallel$. Usually it is assumed that the dynamics is self-affine, which implies that $\nu_\perp=\chi\nu_\parallel$ [6,7]. This defines the roughness exponent χ that characterizes the spatial correlations of the interface. If ''simple scaling'' [35,7] holds, then one has a unique roughness exponent and we can write for the interface width w

$$W^2(t,L)\sim\begin{cases}t^{2\beta w} & t\ll t_x\\ L^{2\alpha}, & t\gg t_x\end{cases}, \tag{A2}$$

using also the early-time exponent β_w. If simple scaling holds, we have the exponent relation $\beta_w z=\alpha$ [35]. If only one timescale is present, the growth exponent is related to the activity time-decay exponent, θ, via $\theta+\beta_W=1$ [31].

For point-like disorder the first-loop functional renormalization group result reads $\chi=(4-d)/3$, and $z=2-(4-d)/9$ [14]; see the extension to second order in [16]. From these, using the exponent relations, the other exponents follow. For rather generic bare disorder correlators the implication is that the full correlator flows in the renormalization to this ''random field'' (or point-disorder) fixed point function, and thus the exponents are the same. However, numerics in particular in 1D implies that the real exponents are different from the one-loop results. This has recently been explained in terms of two-loop corrections, but the traditional interpretation has been in terms of ''anomalous scaling'' [17,38], meaning that as $t\to\infty$, the typical height difference between neighboring sites increases without limit.

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